Accommodating Obstacle Avoidance in the Weapons Allocation Problem for Tactical Air Defence

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Abstract - *In the defence domain, weapons allocation is defined to be the reactive assignment of weapon systems to engage or counter identified threats. From a command perspective, this refers to the allocation of friendly and coalition force elements (eg fighter aircraft, frigates etc) to engage or interdict adversaries which are posing threats, not only to themselves, but also to defended areas and high-value assets. In an earlier work, a conceptual rule-based approach to weapons allocation in the air domain was outlined in terms of so-called critical and sorting parameters, which may be used to determine the capability of each friendly airborne interceptor to engage or counter each threat, and to rank the candidate interceptor-threat pairings respectively. An issue of relevance to the evaluation of the (interdependent) parameters of fuel sufficiency, egress safety and time-to-intercept is how to determine the shortest path from a given interceptor to a static or dynamic threat which avoids prohibited areas such as missile engagement zones, neutral and enemy territories and other exclusion zones. However, in general finding the shortest path is a non-trivial exercise and so determining suboptimal paths through the prohibited areas is often necessary. In the current paper, the problem is investigated from both perspectives. In particular, a technique developed for, and applied to, the field of robotics for finding the shortest path from a source to a fixed destination through a flat earth environment littered with obstacles is adapted to solve the shortest path problem for a spherical earth geometry. This is then used as the basis for determining efficient paths from an interceptor to engage or counter a moving threat.*

Keywords: Weapons allocation, threat evaluation, air defence, Dijkstra's algorithm, visibility graphs.

1 Introduction

In the defence domain, weapons allocation refers to the reactive assignment of weapon systems to engage or counter identified threats. From a single platform perspective, this typically refers to the allocation of the platform's armaments (eg missiles, bombs, torpedoes etc) or countermeasures (eg chaff, flares, decoys etc) to engage or evade the various threats facing it (see for example [1]). From a command perspective, however, this refers more to the allocation of friendly and coalition force elements (eg fighter air-craft, frigates etc) to engage or interdict adversaries which are posing threats, not only to themselves and other force elements, but also to defended areas and high-value ground and surface-based assets. It is the latter perspective which is relevant to the current paper.

In an earlier work [2], a conceptual rule-based approach to weapons allocation in the air domain has been outlined in terms of so-called *critical* and *sorting* parameters, which may be used to determine the capability of each friendly airborne interceptor to engage or counter each threat, and to rank the candidate interceptor-threat pairings respectively. It lists the critical parameters as:

- Threat priority - This determines the order in which threats are dealt with from the most to the least threatening (if they are to be handled sequentially);

- Interceptor availability - This indicates whether the interceptor is available for allocation to a threat or if instead it has already been assigned a task;

- Interceptor suitability - This indicates whether the platform is of the correct category eg fighter or bomber, for allocation to a threat;

- Weapons effectiveness - This determines whether the interceptor's weapons are capable of defeating the threat;

- Fuel sufficiency - This determines if the interceptor has sufficient fuel to reach the threat, engage or counter it and then return to base or force; and

- Egress safety - This determines if the interceptor will have an unobstructed path on its return to base or force, after having engaged or countered the threat,

while the sorting parameters are:

- Time-to-intercept[1] - This is an estimate of the time required for the interceptor to come into range of the threat to engage or counter it;

- Intercept path - This determines whether the interceptor may intercept the threat without having to pass through hostile territory or other prohibited areas; and

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[1] Arguably, time-to-intercept should also be used as a critical parameter, since an interceptor should not be considered for pairing with a threat if it can't reach it before it carries out its mission.

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- Intercept angle - Based on the required nature of the intercept, this determines if the intercept geometry is suitable (eg collision-collision intercepts for engagements and pure pursuit intercepts for identification).

An issue of relevance to the evaluation of the (interdependent) parameters of fuel sufficiency, egress safety and time-to-intercept is how to determine the shortest path from a given interceptor to a static or dynamic threat which avoids prohibited areas such as missile engagement zones, neutral territories and other exclusion zones. However, in general, finding the shortest path is a non-trivial exercise and so determining suboptimal paths through the prohibited areas which are efficient in terms of the distance travelled and the time to earliest weapon release is often necessary. In the current paper, the problem is investigated from both perspectives.

In particular, a motion planning technique developed for, and applied to, the field of robotics for finding the shortest path from a source to a fixed destination through a flat earth environment littered with obstacles is adapted to solve the shortest path problem for a spherical earth geometry. This is then used as the basis for determining the shortest path from the interceptor to engage or counter a static threat and an efficient path from an interceptor to engage or counter a moving threat.

The remainder of the paper is structured as follows. In Section 2, the threat interception problem is formulated. In addition, relevant mathematical notation, terminology and results are introduced. In Section 3, the threat interception problem in the absence of prohibited areas is discussed for a spherical earth geometry. In Section 4, the so-called shortest path problem is discussed in detail. In Section 5, the solution to the shortest path problem is used to modify the approach to the threat interception problem to allow for the presence of prohibited areas. In Section 6, a brief example is presented to illustrate the proposed techniques. Finally, some concluding remarks are made on possible avenues for further research.

2 Problem Formulation and Mathematical Details

In the context of tactical air defence, consider a scenario in which a threat evaluation has determined that a given threat T is to be intercepted. Furthermore, suppose that a friendly interceptor I is a candidate for the interception. In evaluating the pairing of I to T, it needs to be determined if I is capable of intercepting T, that is bringing T within its weapons envelope, and if so, what the minimum time to the intercept (also know as the time to earliest weapon release) is and what path I should follow to achieve the interception within this minimum time. Allowance also needs to be made for the fact that I is to avoid flying through the airspace of nominated prohibited areas such as missile engagement zones, neutral and enemy territories and other exclusion zones for example. A depiction of such a scenario with respect to a spherical earth geometry appears in Fig. 1, featuring the interceptor on the left in blue, the threat (and its initial heading) on the right in red, and the prohibited areas between them in magenta.
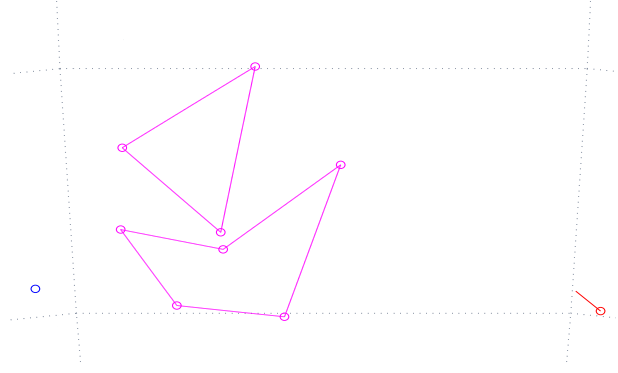

Figure 1: A Threat Interception Scenario Involving Prohibited Polygonal-Shaped Areas.

In determining if I is capable of intercepting T, the following simplifying assumptions are made[2]:

1. The interceptor I and the threat T are considered to be point objects flying in the airspace above a spherical earth with centre O and radius R_e;

2. I and T fly level at the same constant altitude h and at the constant speeds $v_I = ||\boldsymbol{v}_I(t)||$ and $v_T = ||\boldsymbol{v}_T(t)||$ respectively. Moreover, the initial position $P_T(0)$ and velocity $\boldsymbol{v}_T(0)$ of T are known, but only the initial position $P_I(0)$ and speed of I are known (the initial direction of I is to be determined as part of the solution to the intercept problem);

3. The maximum weapons range of I is R_W and the time to earliest weapons release for I is the first instant that I comes within great circle distance R_W of T;

4. T flies along the great circle path (trajectory) determined by its initial position and velocity. Also, the position of the intercept (if it is possible) and the position of I at all times lie at a great circle distance strictly less than $\pi(R_e + h)/2$ units of $P_T(0)$;

5. I is to intercept T in the minimum time possible. It is noted that since I is assumed to fly at constant speed, this means equivalently that I is assumed to intercept T along the shortest path possible; and

6. With the convention that a point in the airspace *lies above* an area on the earth's surface if and only if its projection onto the earth's surface in the direction of O lies in the area, it assumed that each prohibited area may be modelled as the block of airspace which lies above the interior of a spherical polygon on the earth's surface. Furthermore, the prohibited areas are assumed to be pairwise disjoint.

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[2]While the assumptions have been made either to assist in formulating the problem or to make it mathematically tractable, it is remarked that none of the assumptions is unreasonable and so they are not so restrictive as to produce misleading results. The calculations, which are made at each time instant, are intended to support decision-making by defence operators who would be monitoring the situation as it unfolds and so would be cognisant of any substantial changes to the situation that might invalidate the assumptions or necessitate a new evaluation.

2.1 Mathematical Preliminaries

Throughout the remainder of the paper, the following mathematical notation, terminology and results are employed.

Unless otherwise specified, standard Euclidean coordinates, also referred to as earth-centred rotating (ECR) coordinates [3, p. 176], are used to represent points, vectors, planes and all other geometrical entities. Vectors are written either as lower case letters in boldface such as $\mathbf{v}$ or in the form $\overrightarrow{PQ}$ if they specify the vector pointing in the direction of point Q from point P. In particular, the position vector of the point P is denoted by $\overrightarrow{OP}$ and is occasionally identified with P for the sake of convenience. For all non-zero vectors $\mathbf{v}$, the unique unit vector in the direction of $\mathbf{v}$ will be denoted by $\check{\mathbf{v}}$. All other standard vector notation, such as $||\mathbf{v}||$ for the length of $\mathbf{v}$, is also adhered to.

A continuous function $f : [a, b] \rightarrow \mathbb{R}^3$ is called a *path* in $\mathbb{R}^3$ from $f(a)$ to $f(b)$ [4, p. 59]. Unless specified otherwise, all paths in this paper are assumed to be smooth almost everywhere and so have a well-defined length. Given two non-diametrically opposed points P and Q on a sphere of radius R and centre O, the *great circle* defined by P and Q is the intersection of the sphere with the plane determined by P, Q and O. For two such points P and Q, the shorter of the two segments on the great circle from P to Q is the *great circle path* from P to Q and is denoted by $\overset{\frown}{PQ}$. It can be shown that the great circle path from P to Q is also the shortest path on the sphere's surface from P to Q. The great circle distance between P and Q is denoted by $||\overset{\frown}{PQ}||$ and is related to the Euclidean distance between P and Q according to the equation

$$||\overset{\frown}{PQ}|| = 2R \arcsin(||\overrightarrow{PQ}||/2R). \tag{1}$$

Furthermore, setting $\mathbf{n} = ||\overset{\smile}{OP}||$, $\mathbf{t} = ||\check{\mathbf{v}}||$, where $\mathbf{v} = (\overrightarrow{OP} \times \overrightarrow{OQ}) \times \overrightarrow{OP}$, and $d = ||\overset{\frown}{PQ}||$, a general point on the great circle path between P and Q may be represented in parametric form as $P(s)$ such that

$$\overrightarrow{OP}(s) = R \cos(s/R)\mathbf{n} + R \sin(s/R)\mathbf{t}, \tag{2}$$

where $s \in [0, d]$ denotes the arc length from P to $P(s)$ along the great circle path (see [5, Eq. 58]). It is also noted that $\mathbf{n}$ is normal to the sphere at P and that $\mathbf{t}$ is tangent to the sphere at P in the direction of Q. A path on a sphere is said to be *polygonal* if and only if it consists of a finite number of great circle paths such that no two connected great circle paths lie on a common great circle. The endpoints of the great circle paths other than the points $f(a)$ and $f(b)$ are referred to as the *inner vertices* of the path. A *spherical polygon* is a simple closed polygonal path on a sphere (that is, a simple polygonal path for which $f(b) = f(a)$).

A *graph* $\mathcal{G}$ consists of a finite set $\mathcal{V}$ and a finite multiset $\mathcal{E}$ consisting of two-element multisets from $\mathcal{V}$ [6, p. 197]. The elements of $\mathcal{V}$ are called *vertices* and those of $\mathcal{E}$ are called *edges*. If an edge e contains a vertex v, then e is said to be *incident* with v and v is said to be an *endpoint* of e. The vertices contained in a given edge are said to be *adjacent*. Any edge which has the same vertex for both its endpoints is called a *loop* and any two vertices are called *multiply connected* if they are adjacent with respect to at least two distinct edges. Any graph without loops and multiply connected vertices is called *simple*. If the elements of $\mathcal{E}$ are ordered pairs, then $\mathcal{G}$ is said to be *directed*, otherwise the graph is said to be *undirected*. If each edge has a real number (a *weight*) assigned to it, then $\mathcal{G}$ is said to be a *weighted* graph. Finally, a sequence of vertices and edges $v_1, \{v_1, v_2\}, v_2, \{v_2, v_3\}, v_3, \ldots, v_{n-1}, \{v_{n-1}, v_n\}, v_n$ for which all the vertices are distinct is called a $\mathcal{G}$-*path* from v_1 to v_n [3].

3 Unobstructed Threat Interception

As a first step towards solving the complete threat interception problem, means of solving the problem in the absence of prohibited areas are investigated. Throughout this section, it is assumed that the threat is not initially within weapons range of the interceptor.

Given the assumptions on the dynamic behaviour and attributes of the threat T and the interceptor I, the unobstructed threat interception problem is straightforward to formulate. Since intercepting T, if it is possible, is interpreted as bringing I within distance R_W of T in the minimum time possible, it follows that at the time t^* of the intercept the great circle distance between T and $P_I(0)$ will be equal to the distance travelled by I plus its weapons radius, that is

$$||P_I(0)\overset{\frown}{P_T}(t^*)|| = v_I t^* + R_W. \tag{3}$$

Thus, to solve the intercept problem, it is sufficient to determine the smallest positive number t^* satisfying Eq. 3. Once t^* is found, the final position of T can be recovered via Eq. 2 as

$$\overrightarrow{OP_T}(t^*) = R \cos(v_T t^*/R)\mathbf{n}_1 + R \sin(v_T t^*/R)\mathbf{t}_1 \tag{4}$$

where $R = R_e + h$, $\mathbf{n}_1 = \overset{\smile}{OP_T}(0)$ and $\mathbf{t}_1 = \check{\mathbf{v}}_T(0)$. This in turn allows the initial velocity of I to be determined as

$$\mathbf{v}_T(0) = v_I \check{\mathbf{u}}, \tag{5}$$

where

$$\mathbf{u} = (\overrightarrow{OP_I}(0) \times \overrightarrow{OP_T}(t^*)) \times \overrightarrow{OP_I}(0), \tag{6}$$

and the final position of I to be determined as

$$\overrightarrow{OP_I}(t^*) = R \cos(v_I t^*/R)\mathbf{n}_2 + R \sin(v_I t^*/R)\mathbf{t}_2 \tag{7}$$

where $\mathbf{n}_2 = \overset{\smile}{OP_I}(0)$, $\mathbf{t}_2 = \check{\mathbf{v}}_I(0) = \check{\mathbf{u}}$ and $R = R_e + h$ as before.

Unfortunately, it is impossible to solve Eq. 3 in closed form in general. However, fixed-point algorithms such as the Newton-Raphson method [7, p. 79] may be readily applied to find t^* recursively or determine that no such such t^* exists, as the case may be.

A second approach, which is only valid over distances for which constant velocity motion is a good approximation to motion along a great circle path at constant speed, is to

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[3]It is more customary to refer to such a sequence simply as a path from v_1 to v_n. However, the term $\mathcal{G}$-path is used herein to avoid any possible confusion between a path in the graphical sense and a path in $\mathbb{R}^3$.

assume that T and I both travel at constant velocity. Under this assumption, Eq. 3 may be replaced by the equation

$$|| \overrightarrow{P_I(0)P_T}(t^*) || = v_I t^* + R_W, \qquad (8)$$

where T is now taken to be within weapons range of I whenever it is within Euclidean distance R_W of I. Unlike Eq. 3, Eq. 8 is straightforward to solve in closed form. On expanding and rearranging the terms in Eq. 8, a quadratic equation $a(t^*)^2 + bt^* + c$ is obtained with

$$
\begin{aligned}
a &= v_I^2 - v_T^2 & (9) \\
b &= 2(R_W v_I + \boldsymbol{v}_T \cdot (\overrightarrow{OP_I}(0) - \overrightarrow{OP_T}(0)) \\
c &= (R_W^2 - || \overrightarrow{OP_I}(0) - \overrightarrow{OP_T}(0) ||^2).
\end{aligned}
$$

A detailed analysis of the quadratic yields the following solutions for the smallest positive value of t^*, if it exists:

$$
t^* = \begin{cases}
-c/b & \text{if } a = 0 \text{ and } b > 0, \\
(-b + \Delta^{1/2})/2a & \text{if } a < 0, b > 0 \\
& \text{and } \Delta \geq 0, \text{ or if } a > 0, \\
\text{is undefined} & \text{otherwise,}
\end{cases} \qquad (10)
$$

where $\Delta = b^2 - 4ac$ denotes the discriminant of the quadratic. Whenever t^* exists, the final position of T, the velocity of I and the final position of I are respectively:

$$
\begin{aligned}
\overrightarrow{OP_T}(t^*) &= \overrightarrow{OP_T}(0) + \boldsymbol{v}_T t^*, & (11) \\
\boldsymbol{v}_I &= v_I \overrightarrow{P_I(0)P_T}(t^*), & (12) \\
\overrightarrow{OP_I}(t^*) &= \overrightarrow{OP_I}(0) + \boldsymbol{v}_I t^*. & (13)
\end{aligned}
$$

Otherwise, the intercept is impossible and so these terms are undefined.

4 Shortest Path Problem

In a range of motion planning problems, it is often of interest to determine the shortest path that allows an object to move from a given source or point of origin to a given destination, whilst avoiding particular obstacles that exist in the object's environment [8, 9, 10]. For example, in robotics, a common problem is that of determining how to plan the shortest path for a robot to follow that avoids collisions with the objects in the room, factory or terrain etcetera in which it is operating. Different assumptions about the nature of the object, the models of the obstacles and the properties of the desired shortest path have led to the development of a variety of approaches and treatments of the problem in the plane (see [8, 9, 10, 11] and [12] for example). The approach of relevance for this paper, given the assumptions that the threat and interceptor are point targets and that the prohibited areas are based on (spherical) polygons, is that which employs the notion of a visibility graph and Dijkstra's algorithm for finding the minimum weight $\mathcal{G}$-path (min $\mathcal{G}$-path) in a weighted graph [8, 9].

4.1 Visibility with Respect to a Planar (Flat Earth) Geometry

In the context of the shortest path problem in the plane, the notion of *visibility* arises naturally through the need to determine if two points A and B are connected by a line segment that does not intersect any of the obstacles. In general, points A and B are defined to be *visible* to each other with respect to a given set S if no point of S lies on the line segment $\overline{AB} \setminus \{A, B\}$. To facilitate the solution of the shortest path problem in the plane involving polygonal-shaped obstacles, in which A and B are the point of origin and the destination, a graph known as the *visibility graph* may be defined. The vertex set of the visibility graph comprises the two points A and B and the vertices of the polygonal obstacles. Two points of the visibility graph are then adjacent by definition if and only if they are visible to each other with respect to the obstacles and the other vertices when regarded as points in the plane. The visibility graph may be used to construct a *weighted visibility graph* by attaching a weight to each edge equal in value to the Euclidean distance between the two endpoints (regarded as points in the plane) of the edge. It will be seen in Section 4.4 that this notion of visibility can be straightforwardly generalised for solving the shortest path problem on a sphere. For example, the visibility graph based on the interceptor's initial position and the set of vertices of the polygons in Fig. 1 is illustrated in Fig. 2.
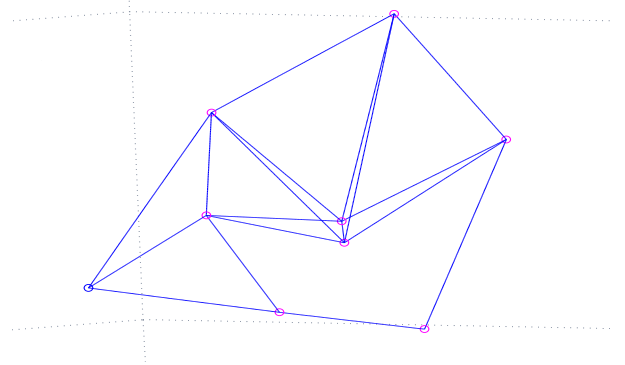

Figure 2: The Visibility Graph Based on the Interceptor's Initial Position and the Polygon Vertices for the Target Interception Example in Fig. 1.

The significance of the visibility and weighted visibility graphs to the shortest path problem in the plane is evident through the following theorem.

Theorem 1: [9, p. 309] *Any shortest path between A and B among a set of disjoint polygonal obstacles is a polygonal path whose inner vertices are vertices of the obstacles.*

Theorem 1 implies that the shortest path between A and B which avoids the obstacles consists of a sequence of line segments whose endpoints (other than A and B) are (i) vertices of the obstacles and (ii) visible to each other. Hence, the shortest path in the plane between A and B corresponds to the min $\mathcal{G}$-path in the weighted visibility graph.

A number of algorithms of differing computational complexity exist for constructing a visibility graph. The naive approach to the problem is to cycle through each pair of vertices and determine their visibility with respect to the edges of the obstacles. If the number of vertices is n, then this approach leads to an algorithm that runs in $O(n^3)$ time since the number of pairs of vertices is $O(n^2)$ and the number of edges is $O(n)$ (see [8, p. 297] and [9, p. 310]). Less

computationally expensive algorithms exist [13], but for the purposes of the paper, the naive approach is sufficient. For a more in depth treatment of visibility in the plane, refer to Chapter 19 of Ref. [10, pp. 829-876].

4.2 Dijkstra's Algorithm

Having constructed the weighted visibility graph, it still remains to determine the min $\mathcal{G}$-path from the given vertex A to the other given vertex B to find the shortest path from A to B. One well-known and often applied algorithm for determining the min $\mathcal{G}$-path between two vertices is Dijkstra's algorithm, which was discovered by the Dutch computer scientist Edsger Dijkstra in 1959 [8]. In fact, Dijkstra's algorithm determines the min $\mathcal{G}$-path from the first given vertex A to every other vertex in the graph. Variants of Dijkstra's algorithm exist for directed and undirected graphs as well as for simple and multiply-connected graphs. However, the variant of relevance here is that which applies to simple, undirected graphs since weighted visibility graphs fall into this class. The description of Dijkstra's algorithm presented below has been sourced from Ref. [14].

For convenience, let the vertex set of the weighted graph $\mathcal{G}$ be $\mathcal{V} = \{1, \ldots, n\}$ with 1 as the given vertex, and for each i, j in $\mathcal{V}$ set $W(i, j)$ equal to 0 if $i = j$, to the weight of the edge $\{i, j\}$ if i and j are adjacent (in which case they are necessarily distinct) and to ∞ otherwise. Then:

Dijkstra's Algorithm

Set $L = \{1\}$

For $i = 1, \ldots, n$

    Set $D(i) = W(1, i)$

    If $W(1, i) = \infty$, set $P(i) = 0$

    Otherwise set $P(i) = 1$

End for

While $\mathcal{V} \setminus L \neq \varnothing$

    Choose $k \in \mathcal{V} \setminus L$ with $D(k)$ as small as possible

    Put k in L

    For each $j \in \mathcal{V} \setminus L$

        If $D(j) > D(k) + W(k, j)$

            Replace $D(j)$ by $D(k) + W(k, j)$

            Replace $P(j)$ by k

    End for

End while

End

At the completion of the algorithm, the quantity $D(j)$ is the length of the min $\mathcal{G}$-path from 1 to j (where $D(j) = \infty$ indicates that no path exists between 1 and j). To recover the min $\mathcal{G}$-path itself, the function P can be used. If $P(j) = 0$, then no path exists between 1 and j. Otherwise, the sequence

$$j, P(j), P(P(j)), P(P(P(j))), \ldots \qquad (14)$$

lists the vertices on a min $\mathcal{G}$-path from 1 to j in reverse order.

For example, consider the visibility graph $\mathcal{G}$ depicted in Fig. 2. Representing its adjacency matrix as

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad (15)$$

where the first row and column correspond to the interceptor's initial position, the weighted graph $\mathcal{G}_W$ may be represented by

$$\begin{bmatrix} 0 & 321.7 & 417.6 & \infty & \infty & \infty & \infty & 543.8 & \infty \\ 321.7 & 0 & 305.9 & \infty & \infty & 306.9 & 292.7 & 278.3 & \infty \\ 417.6 & 305.9 & 0 & 316.3 & \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 316.3 & 0 & 542.7 & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & 542.7 & 0 & 447.5 & 418.5 & \infty & 417.3 \\ \infty & 306.9 & \infty & \infty & 447.5 & 0 & 58.4 & 454.2 & 629.0 \\ \infty & 292.7 & \infty & \infty & 418.5 & 58.4 & 0 & 406.8 & 572.9 \\ 543.8 & 278.3 & \infty & \infty & \infty & 454.2 & 406.8 & 0 & 476.4 \\ \infty & \infty & \infty & \infty & 417.3 & 629.0 & 572.9 & 476.4 & 0 \end{bmatrix}. \qquad (16)$$

Applying Dijkstra's algorithm to the weighted visibility graph $\mathcal{G}_W$ above yields the distance matrix D and the path matrix P respectively given by:

j	$D(j)$	$P(j)$
1	0	1
2	321.7	1
3	417.6	1
4	733.9	3
5	1032.9	7
6	628.7	2
7	614.4	2
8	543.8	1
9	1020.2	8

$$(17)$$

Hence, for example, the shortest distance between vertex 1 (the interceptor) and vertex 5 is $D(5) = 1032.9$ and the $\mathcal{G}$-path along which it is attained may be recovered in reverse order as $5, P(5) = 7, P(7) = 2, P(2) = 1$, that is the $\mathcal{G}$-path is $1, 2, 7, 5$.

4.3 Shortest Path Problem on a Sphere

To solve the shortest path problem on a sphere of radius R and centre O, in which the object in question is restricted to move along a piecewise smooth path on the sphere, the same approach used for the planar case may be re-used with minor modifications primarily to the notion of visibility. Since the shortest (piecewise) path between two (non-diametrically opposed) points on the sphere is the great circle path between them and the obstacles are spherical polygons, two (non-diametrically opposed) points A and B on the sphere are said to be visible to each other with respect to a set S if and only if no point of S lies on the great circle path $\overset{\frown}{AB} \setminus \{A, B\}$. The concepts of visibility graphs and weighted visibility graphs are then defined exactly as for the planar case, and so Dijkstra's algorithm also remains valid for determining the min $\mathcal{G}$-path. However, what underpins the generalisation of the planar approach to the spherical case is the fact that Theorem 1 remains valid if "polygonal" is replaced with "spherical polygonal", that is the following theorem holds.

Theorem 2: *Any shortest path between A and B among a set of disjoint spherical polygonal obstacles is a spherical*

polygonal path whose inner vertices are vertices of the obstacles.

A formal proof of Theorem 2 established by the authors will appear elsewhere. However, it follows the same logic as that employed for the planar case (see [8, pp. 295-297] and [9, p. 309]). Informally, the two steps in the proof may be summarised as follows. First it can be shown that a shortest path cannot contain a smooth subpath other than a great circle path, because such a path can always be shortened by replacing part of it with a great circle path. Hence, the shortest path consists of a sequence of great circle paths. Second it can be shown that any inner vertex of the path must be a vertex of one of the obstacles, because in the neighbourhood of any inner vertex that were not also a vertex of one of the obstacles, the path could be shortened by replacing a portion of it by a suitable great circle path. As for the planar case, Theorem 2 guarantees that the shortest path corresponds to the min $\mathcal{G}$-path in the weighted visibility graph because consecutive inner vertices are visible to each other (this follows from the facts that the shortest path by definition intersects none of the obstacles and the segment of the shortest path between them has been shown to be a great circle path).

4.4 Visibility with Respect to a Spherical Earth Geometry

Since it has been necessary to redefine visibility for the spherical case, it remains to demonstrate means of determining the visibility graph. To exploit the existing algorithms for the planar approach, the approach proposed herein is to project each point of the set S, which comprises the point of origin, the destination and all the vertices of the obstacles, onto a selected tangent plane of the sphere in such a way that visibility is preserved. After introducing local coordinates into the plane, existing algorithms for the planar case may be employed to construct the visibility graph. The details follow.

Projection

To systematically choose a tangent plane into which to project the points, all that is needed is a suitable choice for the point of tangency. One solution is to calculate the barycentre P_b of the vertices of the obstacles and use its projection onto the sphere in the direction of the vector $\overrightarrow{OP_b}$ as the point of tangency τ. Given m obstacles such that the ith obstacle has n_i vertices v_{ij}, the barycentre P_b is given by

$$P_b = \sum_{i=1}^{m} \sum_{j=1}^{n_i} \mathrm{v}_{ij} / \sum_{i=1}^{m} n_i \qquad (18)$$

and so the point of tangency τ is given by

$$\overrightarrow{O\tau} = R \, \overrightarrow{\breve{OP_b}}. \qquad (19)$$

Then each point P of S is projected onto the plane in the direction of the vector $\overrightarrow{OP}$. Given that the equation of the tangent plane at τ is given by

$$(\boldsymbol{x} - \overrightarrow{O\tau}) \cdot \overrightarrow{O\tau} = 0 \qquad (20)$$

a simple calculation yields the projection P' of P onto the plane as

$$\overrightarrow{OP'} = (R^2/(\overrightarrow{O\tau} \cdot \overrightarrow{OP})) \, \overrightarrow{OP}. \qquad (21)$$

It is noted that the projection used maps the great circle paths between the points of S to the corresponding line segments between the projections of the points onto the tangent plane. Therefore two points in S are visible to each other on the sphere if and only if their projections are visible to each other in the tangent plane. In other words, the projection preserves visibility.

Local Coordinates

Most algorithms for determining visibility in the plane rely on algebraic manipulations of the 2-dimensional coordinates of the vertices. Therefore, while it is not essential, it is arguably preferable to transform the coordinates of points in the tangent plane from their 3-dimensional form in ECR coordinates to an essentially 2-dimensional form in local coordinates. If $\tau = (0, 0, \pm R)$, then the tangent plane has equation $z = \pm R$, so the points are already in a suitable form, since the z coordinate of each point is the same and so may be ignored. Otherwise, denote τ by (t_1, t_2, t_3) and define the set $\mathcal{F} = \{\boldsymbol{u}_x, \boldsymbol{u}_y, \boldsymbol{u}_z\}$ where

$$
\begin{aligned}
\boldsymbol{u}_z &= (t_1, \breve{t_2}, t_3) & (22) \\
\boldsymbol{u}_x &= (-t_2, \breve{t_1}, 0) & (23) \\
\boldsymbol{u}_y &= \boldsymbol{u}_z \times \boldsymbol{u}_x. & (24)
\end{aligned}
$$

Since $\boldsymbol{u}_z \neq (0, 0, \pm 1)$, it is straightforward to check that the vectors in $\mathcal{F}$ form an orthonormal basis which can be used to define the axes for a local coordinate system with τ as the local origin. Since $\boldsymbol{u}_z$ is orthogonal to the tangent plane, it follows that $\boldsymbol{u}_x$ and $\boldsymbol{u}_y$ lie in the plane. For any point $\boldsymbol{x} = (x_1, x_2, x_3)$ in the tangent plane, its coordinates with respect to the local coordinates[4] are $[\boldsymbol{x}]_{\mathcal{F}} = (x_1', x_2', x_3')$ where $(\boldsymbol{x} - \overrightarrow{O\tau}) = x_1' \boldsymbol{u}_x + x_2' \boldsymbol{u}_y + x_3' \boldsymbol{u}_z$. Taking dot products of both sides of this last equation with respect to each of the basis vectors in $\mathcal{F}$ gives

$$
\begin{aligned}
x_1' &= (\boldsymbol{x} - \overrightarrow{O\tau}) \cdot \boldsymbol{u}_x & (25) \\
x_2' &= (\boldsymbol{x} - \overrightarrow{O\tau}) \cdot \boldsymbol{u}_y & (26) \\
x_3' &= (\boldsymbol{x} - \overrightarrow{O\tau}) \cdot \boldsymbol{u}_z = 0. & (27)
\end{aligned}
$$

Since the local "z" coordinate is 0 for each point, it may be ignored, leaving an essentially 2-dimensional coordinate system as required.

Determining Visibility

Having reduced the visibility problem on the sphere to the visibility problem in the plane, any existing algorithm for determining visibility may be used to construct the visibility graph. For discussions on how to construct the visibility graph in the planar case, refer to [8] and [9].

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[4]These coordinates are sometimes referred to as East-North-Up coordinates because of the directions that the vectors $\boldsymbol{u}_x, \boldsymbol{u}_y$ and $\boldsymbol{u}_z$ point with respect to a spherical earth (cf. the North-East-Down coordinates described in Ref. [3, p. 176]).

5 Threat Interception in the Presence of Prohibited Areas

The techniques and results from Sections 3 and 4 may now be combined to determine "satisficing" solutions to the moving threat interception problem in an environment containing prohibited areas.

The key to the solution is the observation that if the interceptor I is able to intercept the threat T and that it does so in the shortest time possible, then the final great circle path (leading to the weapon release point) emanates either from the initial position of I or from a vertex of one of the prohibited areas. Thus, the shortest path to intercept T, if one exists, lies in the set S_π of paths constructed as follows. Let S denote the set consisting of the initial position $P_I(0)$ of I and the n vertices of the prohibited areas. For each element V of S, compute the time t_V required by I to travel along the shortest path $\pi_{\min}(V)$ from $P_I(0)$ to V and determine if I can intercept T (now at position $P_T(t_V)$) from V along a path such that T remains visible to I. If so, for each such path, define the augmented path $\pi_{\min}(V) \cup \overset{\frown}{VP_I}(t_V^*)$, where t_V^* is the time to the intercept from V. As V varies over the set S, the union of all such paths $\pi_V = \pi_{\min}(V) \cup \overset{\frown}{VP_I}(t_V^*)$ constitutes the set S_π. Then I is able to intercept T whilst travelling at constant speed $\| \boldsymbol{v}_I \|$ and also avoiding the prohibited areas if and only if S_π is non-empty. The heuristic employed here to find reasonable (suboptimal) solutions is not to consider every path in S_π, but to restrict attention to those paths which are constructed as outlined in the following algorithm. From amongst those paths considered, the path ultimately selected (assuming at least one exists) is the one for which the time to earliest weapon release is minimum.

Initialisation Set the minimum time to intercept (that is, the time to earliest weapon release) to ∞ and the shortest intercept path from $P_I(0)$ to T to $\varnothing$.

Step 1: Given the initial conditions for the threat T and the interceptor I, determine if I can intercept T assuming that the prohibited areas do not exist. If it cannot, then halt the algorithm[5]. Else proceed to step 2.

Step 2: Still under the same assumption, check if the resulting path that I would follow to intercept T passes through any of the prohibited areas (that is, determine the visibility of I's initial position and its final position with respect to the prohibited areas). If the initial and final positions are visible to each other with respect to the polygons, then set the minimum time to intercept to the time required for I to travel along this great circle path, set the shortest intercept path to the great circle path[6] and halt the algorithm. Otherwise, proceed to step 3.

Step 3: Using the techniques described in Section 4, con-

struct the visibility graph $\mathcal{G}$ for the set of vertices S consisting of $P_I(0)$ and the n vertices of the prohibited areas[7].

Step 4: Construct the weighted visibility graph $\mathcal{G}_W$ by associating a weight to each edge of $\mathcal{G}$ that is equal in value to the great circle distance between the corresponding vertices of S. Then apply Dijkstra's algorithm to $\mathcal{G}_W$ to calculate the min $\mathcal{G}_W$-paths from $P_I(0)$ to each of the other vertices and the lengths of these paths.

Step 5: For each vertex $V \in S \setminus \{P_I(0)\}$, if the shortest path from $P_I(0)$ to V is finite:

1. Determine the time t_V required for I to travel to V along the path and determine the position of T at t_V.

2. Determine if I at position V can intercept T (now at position $P_T(t_V)$) at some time t_V^*, under the assumption that there are no prohibited areas. If so, determine if $P_T(t_V^*)$ lies within a great circle distance $\pi R/2$ of $P_T(0)$, and that V and $P_T(t_V^*)$ are visible to each other. If so, then if $(t_V + t_V^*)$ is less than the current minimum time to intercept, reset the minimum time to intercept to $(t_V + t_V^*)$ and the shortest path to $\pi_V = \pi_{\min}(V) \cup \overset{\frown}{VP_I}(t_V^*)$.

Once all the steps have been completed, if the minimum time to intercept is infinite, then I is not able to intercept T using the adopted strategy. Otherwise, I is able to intercept T and the algorithm returns an efficient path from $P_I(0)$ to the point of intercept and the time to earliest weapon release along this path.

6 Example

Consider the interception scenario illustrated in Fig. 1. The interceptor's speed was 1000 km/hr and it had a weapons radius of 30 km. The threat's speed was 700 km/hr. At step 1, using the first unobstructed threat interception technique described in Section 3 resulted in a potential interception solution. However, at step 2, the initial and final interceptor positions were found not to be visible to each other with respect to the polygonal prohibited areas, and so the algorithm proceeded to steps 3 and 4. At these steps, the visibility graph $\mathcal{G}$ and the weighted visibility graph $\mathcal{G}_W$ in Eqs. 15 and 16 were constructed, and Dijkstra's algorithm was employed to find the shortest distances and the associated $\mathcal{G}_W$-paths in Eq. 17 from the initial interceptor position to each vertex of the prohibited (polygonal) areas. Finally, at step 5, exactly two threat interception solutions resulted. The first path proposed was 1, 2, 7, 5 to the threat and the second path proposed was 1, 8, 9 to the threat. The first path was the shorter of the two, so the output of the algorithm was the path 1, 2, 7, 5 to the threat indicated in green in Fig. 3 with a time to earliest weapon release of 8.43 minutes (the figure also indicates the strike envelope of the

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[5]If I cannot intercept T under this assumption, then it will not be able to intercept T in the presence of the prohibited areas either because any path that I were to take around the prohibited areas would necessarily be longer than the great circle path between T and I.

[6]In this case, I can intercept T along a direct great circle path, so that the prohibited areas provide no obstacle. Since any other path around the prohibited areas would necessarily be longer than the great circle path, the shortest intercept path in this case has already been found.

[7]Note that it is not necessary to construct the whole visibility graph from scratch at each iteration. Instead, the (visibility) subgraph of $\mathcal{G}$ based on just the vertices of the prohibited areas can be constructed once, during the initialisation stage for example. Then, at each iteration, it is only necessary to determine the visibility of $P_I(0)$ to each of the vertices of the prohibited areas to complete the subgraph to $\mathcal{G}$. This substantially reduces the computational expense of constructing $\mathcal{G}$.

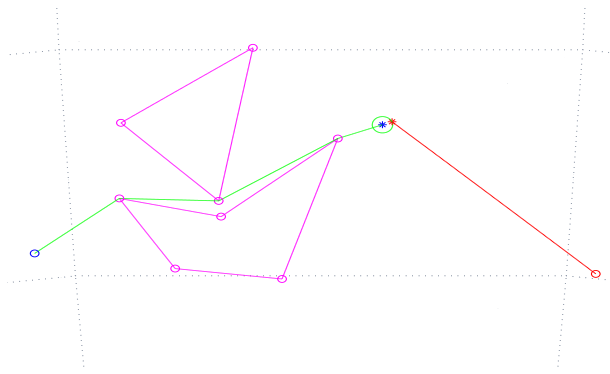interceptor by the green circle and the threat's position at the time of the intercept by the red asterisk).

Figure 3: The Most Efficient Path from the Interceptor to the Threat as Determined by the Algorithm.

7 Conclusion

In the context of weapons allocation for tactical air defence, the problem of threat interception in an environment containing prohibited areas has been investigated and an approach based on techniques from the discipline of computational geometry has been proposed. In particular, by interpreting threat interception as a shortest path problem, solutions have been developed for intercepting static and moving threats by extending a known approach to the shortest path problem in the plane, based on visibility graphs and Dijkstra's algorithm. For static threats, the solution based on the generalisation of the notion of visibility for a spherical earth geometry yields the shortest path from the interceptor to the threat, while for moving threats, the approach described is "satisficing" in nature, producing suboptimal, yet efficient solutions. An example has also been given to illustrate the ideas developed herein. Finally, it is noted that while this problem has been discussed in isolation of the threat evaluation process which led to determining that the threat should be intercepted, the two processes are better regarded as different interdependent aspects of a single threat evaluation and weapons allocation (TEWA) process [15, 16]. As such, future work will focus on incorporating this threat interception technique into a TEWA testbed for the purpose of experimenting with emerging TEWA concepts, as well as testing and evaluating different threat interception strategies.

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